論文

Generalized Plane Deformations of a Laminated Composite Strip Containing a Delamination Crack

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층간균열을 갖는 적층 복합재료 스트립의 일반평면변형

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ABSTRACT

Based upon Lekhnitskii's formulation and Stroh formalism for plane elasticity theory of an anisotropic body, the asympototic solution is examined for a delamination crack in a laminated composite strip undergoing generalized plane deformation under extension, bending and/or torsion. The near-field conditions for the opened and the closed delamination crack are imposed, together with the eigenfunction expansion for the displacement potential, to lead to the structure of solutions which consist of homogeneous solution and particular solution. It appears that no logarithmic solutions exist, regardless of the ply orientations, for each of an opened and a closed crack, and thus power type homogeneous solution and the polynomial type particular solution turn out to be valid.

초 록

인장, 굽힘, 비틀림을 받고있는 층간균열을 갖는 적층복합 스트립이 일반 평면 변형상태에 있을 때 평면탄성이론에 기초한 비등방성 물체를 해석하는데 널리 사용되어지는 Lekhnitskii수식화와 Stroh 수식화를 사용하여 점근해가 검토되어진다. 점근해를 구하기 위해 먼저 변위장에 대한 고유함수 전개를 사용하고 층간균열이 열린 경우와 닫힌 두가지 경우에 대해 near-field 경계조건을 사용하여 점근해의 구조를 얻을 수 있다. 이 점근해는 균질해와 특이해로 구성되어진다. 위의 두가지 층간균열에 대해 모든 적층순서에 관계없이 점근해는 대수함수의 형태를 가지지 않고 멱급수형태의 균질해와 다항식형태의 특이해만으로 나타내어진다.

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1. Introduction

Deformations of a laminated composite inherently involve a boundary layer region on which deformation field is locally distorted owing to material of geometric discontinuity such as ply interfaces, free edges, cracks or cutouts. Generally, the difficulties involve local stress singularities and inherently three-dimensional state of complex stresses. Moreover, the high local stresses and associated deformations caused by these geometric and material discontinuities always result in undesirable delamination and transverse crack initiation and growth, leading to the final fracture. Therefore, development of an analytical method that can provide insight into a boundary layer region under general loadings is of foremost important to the analyst. The interfacial or transverse crack problems and free edge problems are among typical examples concerning boundary layers in mechanics of composite laminates, and they have been among the subjects under intensive investigation during the last two decades.

Suppose a long laminated composite strip is subjected to lateral tractions or/and end tractions such that stresses and strains on every cross section remain unchanged along the length of the strip. Such a deformation is called a generalized plane deformation[1]. Typical examples of this class of deformations include uniform extension, pure bending, torsion and a combination of these modes as well as the well-known generalized plane strain deformations.

From the theory of anisotropic elasticity Lekhnitskii developed complex potentials to treat the generalized plane deformation of an anisotropic strip under general loadings such as uniform extension, bending, torsion or any combination of these. Wang and Choi[2] employed Lekhnitskii's formulation to obtain the solution for a free edge problem under extension. Subsequently Wang[3] obtained the solution for a delamination crack under extension in a similar method. In these works, the particular solution for stresses was assumed to be polynomial functions in the beginning, finally reduced to a constant for uniform extension. On the other hand, Zwiers et al.[4] found, based upon the Stroh formalism[5], that there may exist the logarithmic particular solution for stress in free edge problem under uniform extension. This result indicates that the assumption that a particular solution for stress takes a polynomial functions under complex general loadings other than simple extension is an open questions. Unfortunately, no complete solution other than uniform extension has been reported up to date, although the solution is of paramount importance in association with the understanding of fundamental fracture behavior of composite laminates under more general loadings.

In this paper, we examine the asymptotic solution near the delamination crack tip in a laminated composite strip subjected to the general loadings such as extension, bending and torsion. For this, we rely upon Lekhnitskii's formulation and Stroh formalism, which have been found to be useful for calculating the asymptotic solution for the free edge problem under uniform extension[1,4]. This solution procedure is here extended to the general cases of loading including bending, torsion, as well as extension.

In section 2.1, the problem under consideration is described, and then based upon Lekhnitskii's formulation and Stroh formalism under generalized plane deformation, the solution form for stress and displacement field is obtained, and the asymptotic solutions from the eigenfunction expansion are presented. In section 2.2, the near-field conditions for the opened and the closed delamination cracks are imposed to lead to the structure of solutions which consist of homogeneous solution and particular solution. The asymptotic form of homogeneous solutions for stress and displacement, including the stress singularity is determined in section 2.3. In section 2.4, existence of the polynomial type particular solution in composite laminates subjected to aforementioned general loadings under generalized plane deformation is investigated, and numerical results and discussion are followed in section 3. Finally concluding remarks are made in section 4.

2. Formulation of the Problem

2-1. Generalized Plane Deformation Problem

Consider a laminated composite subjected to general loadings such as *extension*, *bending* and *torsion* (see Fig. 1). Each ply of the composite laminates lies in a plane parallel to the x_1-x_3 plane, and the ply orientation θ is defined to be the counter-clockwise angle, viewed from the top, that the fiber direction makes with the x_3-axis. We assume that the laminate dimension in the x_3 direction(laminate width) is sufficiently large compared with the laminate thickness so that the laminate is assumed to be in the state of generalized plane deformation on the x_1-x_2 plane under the aforementioned loadings[1].

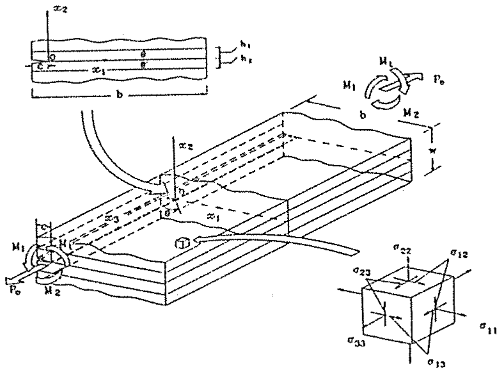

Fig. 1. Delamination crack problem under general loadings such as extension, bending and torsion.

Let u_i, ε_{ij}, σ_{ij} denote the Cartesian components of displacement, strain and stress, respectively. For generalized plane deformation, we have the governing equations:

equilibrium equation:

$$\sigma_{ij,j}=0 \text{(no body force)} \quad \cdots\cdots\cdots\cdots\cdots \quad (1.a)$$

strain-displacement relation:

$$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1.b)$$

stress-strain relation:

$$\sigma_{ij}=C_{ijk\ell}\varepsilon_{k\ell}, \quad C_{ijk\ell}=C_{jik\ell}=C_{k\ell ij} \quad \cdots\cdots\cdots \quad (1.c)$$

where $C_{ijk\ell}$ are 4-th order stiffness tensor, and the comma indicates the partial differentiation with respect to x_i. Introducing the "collapsed" representation, we may write the stress-strain relation as

$$\sigma_i=C_{ij}\varepsilon_j, \quad \varepsilon_i=S_{ij}\sigma_j \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2.a, b)$$

where

$$\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}^T = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\}^T \quad \cdots\cdots\cdots\cdots \quad (2.c)$$

$$\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}^T = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{13}, 2\varepsilon_{12}\}^T \quad \cdots\cdots\cdots\cdots \quad (2.d)$$

and C_{ij} and S_{ij} are the stiffness and the compliance matrix. We supposed the loadings of extension, bending and torsion, only, which result in the state of generalized plane deformation in the composite strip, and make the problem two dimensional. When these loadings are applied to the composite strip, from the constitutive equations and compatibility relation the displacement component u_i may be given as in [1,2]

$$\begin{aligned} u_i(x_1, x_2, x_3) = & \ U_i(x_1, x_2) + \delta_{i1}(-A_2x_3^2/2 - A_4x_2x_3) \\ & + \delta_{i2}(-A_3x_3^2/2 - A_4x_1x_3) \\ & + \delta_{i3}(-A_2x_1 + A_3x_2 + A_1)x_3 \quad \cdots\cdots\cdots \quad (3) \end{aligned}$$

where δ_{ij} is the kronecker delta, and A_1 is parameter related to axial extension along the x_3-axis, A_2 and A_3 to bending in the x_1-x_3 planes and the

x_2-x_3 planes, respectively, and A_4 to torsion along the x_3-axis. For convenience of later development, we write $U_i(x_1, x_2)$ as

$$U_i(x_1, x_2) = \hat{U}_i(x_1, x_2) + \lambda_i x_1^2/2 \quad \cdots\cdots\cdots\cdots (4)$$

where λ_i are constants to be determined from equilibrium equation. The displacement u_i with equation(4) can be rewritten as:

$$\begin{aligned} u_i(x_1, x_2, x_3) &= \hat{U}_i(x_1, x_2) + \lambda_i x_1^2/2 + \\ &\delta_{i1}(-A_2x_3^2/2 - A_4x_2x_3) + \delta_{i2}(-A_3x_3^2/2 - A_4x_1x_3) \\ &+ \delta_{i3}(-A_2x_1 + A_3x_2 + A_1)x_3 \quad \cdots\cdots\cdots\cdots (5) \end{aligned}$$

Substituting equation (5) into equations (1.a)-(1.c), we have

$$\begin{aligned} \sigma_{ij}(x_1, x_2) &= (C_{ij23}A_4 + C_{ij33}A_2)x_1 \\ &+ (C_{ij33}A_3 - C_{ij13}A_4)x_2 \\ &+ C_{ij33}A_1 + C_{ijk1}\lambda_k x_1 \\ &+ [C_{ijk1}\partial\hat{U}_k/\partial x_1 + C_{ijk2}\partial\hat{U}_k/\partial x_2] \quad \cdots (6) \end{aligned}$$

$$\begin{aligned} \sigma_{ij,j} = &(C_{i123}A_4 + C_{i133}A_2) + (C_{i233}A_3 - C_{i213}A_4) \\ &+ C_{i1k1}\lambda_k + [C_{i1k1}\partial^2\hat{U}_k/\partial x_1^2 \\ &+ (C_{i1k2} + C_{i2k1})\partial^2\hat{U}_k/\partial x_1\partial x_2 \\ &+ C_{i2k2}\partial^2\hat{U}_k/\partial x_2^2] = 0 \quad \cdots\cdots\cdots\cdots (7) \end{aligned}$$

From equilibrium equation (7), we obtain the following two sets of equations for λ_k and $\hat{U}_i(x_1, x_2)$:

$$\begin{aligned} &(C_{i123}A_4 + C_{i133}A_2) + (C_{i233}A_3 - C_{i213}A_4) \\ &+ C_{i1k1}\lambda_k = 0 \quad \cdots\cdots\cdots\cdots (8.a) \end{aligned}$$

$$\begin{aligned} &C_{i1k1}\partial^2\hat{U}_k/\partial x_1^2 + (C_{i1k2} + C_{i2k1})\partial^2\hat{U}_k/\partial x_1\partial x_2 \\ &+ C_{i2k2}\partial^2\hat{U}_k/\partial x_2^2] = 0 \quad \cdots\cdots\cdots\cdots (8.b) \end{aligned}$$

We can determine λ_i related to A_2, A_3 and A_4 from equation(8.a). Following Stroh[5], we can show that the general solution of equation (8.b) takes the form as

$$\begin{aligned} &\hat{U}_i(x_1, x_2) = \sum_{k=1}^{6} v_{ik} f(z_k), \\ &z_k = x_1 + \mu_k x_2 \quad (k=1\sim6) \quad \cdots\cdots\cdots\cdots (9) \end{aligned}$$

where μ_k are the eigenvalues to be determined, and $f(z_k)$ is a function of z_k. Substituting equation (9) into equation (8.b), we obtain the equations:

$$D_{ij}(\mu_k)v_{jk} = 0 \quad \cdots\cdots\cdots\cdots (10.a)$$

where

$$D_{ij}(\mu_k) = C_{i1j1} + \mu_k(C_{i1j2} + C_{i2j1}) + \mu_k^2 C_{i2j2} \cdots (10.b)$$

For the existence of nontrivial solutions, we have

$$\det[D_{ij}] = |D_{ij}| = 0 \quad \cdots\cdots\cdots\cdots (11)$$

and the solutions of this sextic equation yield the three pairs of complex conjugate eigenvalues[5]

$$\mu_k = \bar{\mu}_{k+3}, \quad (k=1,2,3) \quad \cdots\cdots\cdots\cdots (12.a)$$

and three pairs of associated eigenvectors v_{ik} can now be obtained from (10.a) through a proper normalization and satisfy the following equation

$$v_{i(k+3)} = \bar{v}_{ik}, \quad (k=1,2,3) \quad \cdots\cdots\cdots\cdots (12.b)$$

In this work we assume that the eigenvalues μ_k are distinct. A discussion of the cases when μ_k has a multiple root can be found in [6]. Equations (5) and (6) for displacement and stress with equation (9) can then be written as

$$u_i(x_1, x_2, x_3) = \sum_{k=1}^{6} v_{ik} f(z_k) + u_i^*(x_1, x_2, x_3) \cdots (13)$$

$$\sigma_{ij}(x_1, x_2) = \sum_{k=1}^{6} \tau_{ijk} \frac{df(z_k)}{dz_k} + \sigma_{ij}^*(x_1, x_2) \quad \cdots (14)$$

where

$$\tau_{ijk} = (C_{ijl1} + \mu_k C_{ijl2})\, v_{lk}$$
$$\text{(no sum on } k,\ k=1\sim6) \quad \cdots\cdots (15.a)$$

$$\begin{aligned} u_i^*(x_1, x_2, x_3) &= \lambda_i x_1^2/2 + \delta_{i1}(-A_2x_3^2/2 - A_4x_2x_3) \\ &+ \delta_{i2}(-A_3x_3^2/2 + A_4x_1x_3) \\ &+ \delta_{i3}(A_2x_1 + A_3x_2 + A_1)x_3 \quad \cdots\cdots (15.b) \end{aligned}$$

$$\begin{aligned} \sigma_{ij}^*(x_1, x_2) &= \sum_{k=1}^{6} + C_{ijk1}\lambda_k x_1 \\ &+ (C_{ij23}A_4 + C_{ij33}A_2)x_1 + (C_{ij33}A_3 - C_{ij13}A_4)x_2 \\ &+ C_{ij33}A_1 \quad \cdots\cdots\cdots\cdots (15.c) \end{aligned}$$

We assume the power type eigenfunction for $f(z_k)$ as given by [2,7]

$$f(z_k) = \sum_{n=1}^{\infty} C_{kn} z_k^{\delta_n+1}/(\delta_n+1) \quad (k=1\sim6), \qquad \cdots\cdots\cdots\cdots\cdots \ (16)$$

which leads to the expressions for the displacement and stress field

$$u_i(x_1, x_2, x_3) = \hat{U}_i(x_1, x_2) + u_i^*(x_1, x_2, x_3) \qquad \cdots\cdots\cdots\cdots \ (17.a)$$

$$\sigma_{ij}(x_1, x_2) = \hat{\sigma}_{ij}(x_1, x_2) + \sigma_{ij}^*(x_1, x_2) \qquad \cdots\cdots\cdots\cdots \ (17.b)$$

where

$$\hat{U}_i(x_1, x_2) = \sum_{n=1}^{\infty}\sum_{k=1}^{3} [C_{kn} v_{ik} z_k^{\delta_n+1} + C_{(k+3)n} \bar{v}_{ik} \bar{z}_k^{\delta_n+1}]/(1+\delta_n)$$

$$\hat{\sigma}_{ij}(x_1, x_2) = \sum_{n=1}^{\infty}\sum_{k=1}^{3} [C_{kn} \tau_{ijk} z_k^{\delta_n} + C_{(k+3)n} \bar{\tau}_{ijk} \bar{z}_k^{\delta_n}]$$

and an overbar denotes the complex conjugate; C_{kn} and $C_{(k+3)n}$ are complex constants, and the subscript k means the three pairs of eigenvalues. Here the eigenvalues δ_n are to be determined from the so-called "near-field" conditions, including the traction conditions and interface continuity conditions. The coefficient C_{kn} is dependent upon the associated eigenvalues δ_n, and it can be determined within an arbitrary constants when δ_n is obtained.

Using the polar coordinates (r, ϕ, z), z_k may be written as

$$z_k = x_1 + \mu_k x_2 = r\zeta_k \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ (18.a)$$

where

$$\zeta_k = \cos\phi + \mu_k \sin\phi \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ (18.b)$$

Equations (17.a) and (17.b) are then rewritten as follows

$$u_i(x_1, x_2, x_3) = r^{\delta_n+1} \sum_{n=1}^{\infty}\sum_{k=1}^{3} [C_{kn} v_{ik} \zeta_k^{\delta_n+1} + C_{(k+3)n} \bar{v}_{ik} \bar{\zeta}_k^{\delta_n+1}]/(1+\delta_n) + u_i^*(x_1, x_2, x_3) \qquad \cdots\cdots\cdots\cdots \ (19.a)$$

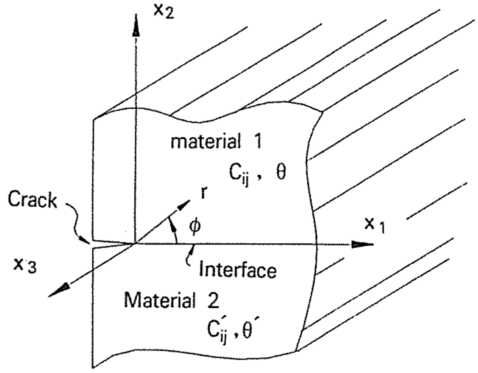

Fig. 2. Delamination Crack in Composite Laminates.

$$\sigma_{ij}(x_1, x_2) = r^{\delta_n} \sum_{n=1}^{\infty}\sum_{k=1}^{3} [C_{kn} \tau_{ijk} \zeta_k^{\delta_n} + C_{(k+3)n} \bar{\tau}_{ijk} \bar{\zeta}_k^{\delta_n}] + \sigma_{ij}^*(x_1, x_2) \qquad \cdots\cdots\cdots (19.b)$$

These expressions may be written for each ply of laminates, for example, the upper ply and the lower ply; in which case we use the above expressions for the upper ply, and the following "primed" expressions for the lower ply(see Fig. 2)

$$u_i'(x_1, x_2, x_3) = r^{\delta_n+1} \sum_{n=1}^{\infty}\sum_{k=1}^{3} [C_{kn}' v_{ik}' \zeta_k'^{\delta_n+1} + C_{(k+3)n}' \bar{v}_{ik}' \bar{\zeta}_k'^{\delta_n+1}]/(1+\delta_n) + u_i'^*(x_1, x_2, x_3) \qquad \cdots\cdots\cdots\cdots \ (20.a)$$

$$\sigma_{ij}'(x_1, x_2) = r^{\delta_n} \sum_{n=1}^{\infty}\sum_{k=1}^{3} [C_{kn}' \tau_{ijk}' \zeta_k'^{\delta_n} + C_{(k+3)n}' \bar{\tau}_{ijk}' \bar{\zeta}_k'^{\delta_n}] + \sigma_{ij}'^*(x_1, x_2) \qquad \cdots\cdots \ (20.b)$$

2-2. Solution Structure under Generalized Plane Deformation

To determine the structure of the asymptotic solution including the stress singularities, we need to consider the near-field conditions. Assuming the two plies are perfectly bonded along the interface, the near-field conditions may be written as :

traction condition:

$\sigma_{12}=\sigma_{22}=\sigma_{23}=0$ at $\phi=\pm\pi$

if crack faces are opened ……… (21)

or

$\sigma_{12}=\sigma_{23}=0$, $[u_2]=[\sigma_{22}]=0$ at $\phi=\pm\pi$

if crack faces are in frictionless contact ………… (22)

interface continuity condition:

$[\sigma_{12}]=[\sigma_{22}]=[\sigma_{23}]=0$

and

$[u_1]=[u_2]=[u_3]=0$ at $\phi=0$ ……… (23)

where [.] indicates the discontinuity of the quantity in it across the ply interface. Substituting equations (19.a) and (19.b) for displacement and stress into near-field conditions, we obtain a system of 12×12 nonhomogeneous linear equations for C_{kn}, $C_{(k+3)n}$, C'_{kn} and $C'_{(k+3)n}$, which can be written as

$$r^{\delta_n}\mathbf{K}_c(\delta_n)\mathbf{q}=A_1\mathbf{b}_1+r(A_2\mathbf{b}_2+A_3\mathbf{b}_3+A_4\mathbf{b}_4) \quad \cdots \quad (24)$$

where $\mathbf{K}_c$ is complex valued square matrix whose elements depend upon δ_n, and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ are constant column matrices related to material constants, and $\mathbf{q}$ is a column matrix whose elements are C_{kn}, $C_{(k+3)n}$, C'_{kn} and $C'_{(k+3)n}(k=1,2,3)$. To satisfy equation (24), we let $\delta_n=0$ and $\delta_n=1$:

$$\mathbf{K}_c(0)\mathbf{q}^{p_1}=A_1\mathbf{b}_1 \quad \cdots\cdots \quad (25.a)$$

$$\mathbf{K}_c(1)\mathbf{q}^{p_2}=A_2\mathbf{b}_2+A_3\mathbf{b}_3+A_4\mathbf{b}_4 \quad \cdots\cdots \quad (25.b)$$

and obtain two particular solutions for $\mathbf{q}$, $\mathbf{q}^{p_1}$ and $\mathbf{q}^{p_2}$. The particular solution $\mathbf{q}^{p_1}$ is related to the parameter A_1, and $\mathbf{q}^{p_2}$ to A_2, A_3 and A_4. This is, however, not the only solution for $\mathbf{q}$ in equation (24). We see that $\mathbf{q}$ has the following homogeneous solution

$$\mathbf{K}_c(\delta_n)\mathbf{q}^h=0 \quad \cdots\cdots \quad (26)$$

Therefore, the complete solution takes a linear superposition of the two:

$$\mathbf{q}=\mathbf{q}^h+\mathbf{q}^p, \quad \mathbf{u}=\mathbf{u}^h+\mathbf{u}^p \quad \cdots\cdots \quad (27.a,b)$$

where

$$\mathbf{q}^p=\mathbf{q}^{p_1}+\mathbf{q}^{p_2}, \quad \mathbf{u}^h=\hat{\mathbf{U}}^h, \quad \mathbf{u}^p=\hat{\mathbf{U}}^p+\mathbf{u}^*.$$

2-3. Homogeneous Solutions

For the existence of nontrivial solution from equation (26), we have

$$|\mathbf{K}_c(\delta_n)|=0 \quad \cdots\cdots \quad (28)$$

which determines the eigenvalues δ_n. When the eigenvalues δ_n are known, within unknown constants the eigenvectors $\mathbf{q}^h$ are computed from equation (26) by a proper normalization. The asymptotic form of homogeneous solutions for the stress and displacement are given by

$$\hat{U}_i^h(x_1,x_2)=u_i^h(x_1,x_2)=\sum_{n=1}^{\infty}\sum_{k=1}^{3}[C_{kn}v_{ik}z_k^{\delta_n+1} + C_{(k+3)n}\bar{v}_{ik}\bar{z}_k^{\delta_n+1}]/(1+\delta_n) \quad \cdots\cdots \quad (29)$$

$$\sigma_{ij}^h(x_1,x_2)=\sum_{n=1}^{\infty}\sum_{k=1}^{3}[C_{kn}\tau_{ijk}z_k^{\delta_n}+C_{(k+3)n}\bar{\tau}_{ijk}\bar{z}_k^{\delta_n}] \quad \cdots\cdots \quad (30)$$

From the structure of $\mathbf{K}_c(\delta_n)$, we can show that if δ_n is a root of the characteristic equation, so is its complex conjugate $\bar{\delta}_n$, and there exists appropriate relation of complex conjugate in C_{kn} so that the expressions (29) and (30) for the stress and displacement become real.

The power type eigenfunction expansion(16) fails to be complete when the algebraic multiplicity is greater than the geometric multiplicity, that is, there are not enough sets of the power type eigenvectors associated with these multiple eigenvalues[6, 8]. Dempsey and Sinclair[8] resolved this difficulty by introducing logarithmic eigenfunctions, which ensure the existence of the sets of eigenvectors enough to span the solution. Subsequently this was extended to the problem of anisotropic composite laminates by Ting and Chou [6]. The existence of the logarithmic eigenfunctions can be examined by calculating the algebraic multiplicity of the eigenvalues and the rank of the associated coefficient matrices in equation(28).

To take only the real part of equations (29) and (30), we may introduce

$$C_{kn}=1/2(\gamma_{1n}-i\gamma_{2n})b_{kn} \text{ for complex } \delta_n, \quad Im[\delta_n]>0$$
$$C_{kn}=1/2(\gamma_{3n}b_{kn}) \text{ for real } \delta_n \quad \cdots\cdots (31.a,b)$$

where b_{kn} is the solution for equation(26), computed by a proper normalization, and γ_{1n}, γ_{2n} and γ_{3n} are normalization constants to be determined to complete the solution. The equations (29) and (30) for displacement and stress are then written as

$$\hat{U}_i^h(x_1,x_2)=u_i^h(x_1,x_2)=\sum_{n=1}^{\infty} Q_{ni},$$
$$\sigma_{ij}^h(x_1,x_2)=\sum_{n=1}^{\infty} P_{nij} \quad \cdots\cdots (32.a,b)$$

where Q_{ni} and P_{nij} are given by :

$$Q_{ni}=\gamma_{1n}Re[\Psi_{ni}]+\gamma_{2n}Im[\Psi_{ni}],$$
$$P_{nij}=\gamma_{1n}Re[\phi_{nij}]+\gamma_{2n}Im[\phi_{nij}]$$
$$\Psi_{ni}=\sum_{k=1}^{3}[b_{kn}v_{ik}z_k^{\delta_n+1}+b_{(k+3)n}\bar{v}_{ik}\bar{z}_k^{\delta_n+1}]/(1+\delta_n)$$
$$\phi_{nij}=\sum_{k=1}^{3}[b_{kn}\tau_{ijk}z_k^{\delta_n}+b_{(k+3)n}\bar{\tau}_{ijk}\bar{z}_k^{\delta_n}]$$

when δ_n is complex;

$$Q_{ni}=\gamma_{3n}Re[\sum_{k=1}^{3}b_{kn}v_{ik}z_k^{\delta_n+1}/(1+\delta_n)]$$
$$P_{nij}=\gamma_{3n}Re[\sum_{k=1}^{3}b_{kn}\tau_{ijk}z_k^{\delta_n}]$$

if δ_n is real.

2-4. Existence of the Polynomial Particular Solutions

To obtain the particular solutions for stress and displacement when δ_n is real, we may choose eigenvector C_{kn} as

$$C_{kn}=\frac{1}{2}(a_{kn}-i\hat{a}_{kn}) \quad \cdots\cdots (33)$$

where a_{kn} and $\hat{a}_{kn}$ are real. Equations (19.a) and (19.b) then have the real expression

$$u_i(x_1,x_2,x_3)=r^{\delta_n+1}\sum_{n=1}^{\infty}\sum_{k=1}^{3}[a_{kn}Re(v_{ik}\zeta_k^{\delta_n+1})+\hat{a}_{kn}Im(v_{ik}\zeta_k^{\delta_n+1})]/(1+\delta_n)+u_i^*(x_1,x_2,x_3) \quad \cdots\cdots (34)$$

$$\sigma_{ij}(x_1,x_2)=r^{\delta_n}\sum_{n=1}^{\infty}\sum_{k=1}^{3}[a_{kn}Re(\tau_{ijk}\zeta_k^{\delta_n})+\hat{a}_{kn}Im(\tau_{ijk}\zeta_k^{\delta_n})]+\sigma_{ij}^*(x_1,x_2) \quad \cdots\cdots (35)$$

For convenience, we set

$$\mathbf{a}^{P_1}=\{a_{10},a_{20},a_{30},\hat{a}_{10},\hat{a}_{20},\hat{a}_{30},a'_{10},a'_{20},a'_{30},\hat{a}'_{10},\hat{a}'_{20},\hat{a}'_{30}\}^T \quad \cdots\cdots (36)$$

$$\mathbf{a}^{P_2}=\{a_{11},a_{21},a_{31},\hat{a}_{11},\hat{a}_{21},\hat{a}_{31},a'_{11},a'_{21},a'_{31},\hat{a}'_{11},\hat{a}'_{21},\hat{a}'_{31}\}^T \quad \cdots\cdots (37)$$

Equations (25.a), (25.b) and (26) are then replaced by

$$\mathbf{K}(0)\mathbf{a}^{P_1}=A_1\mathbf{b}_1 \quad \cdots\cdots (38.a)$$
$$\mathbf{K}(1)\mathbf{a}^{P_2}=A_2\mathbf{b}_2+A_3\mathbf{b}_3+A_4\mathbf{b}_4 \quad \cdots\cdots (38.b)$$
$$\mathbf{K}(\delta_n)\mathbf{a}^h=0 \quad \cdots\cdots (39)$$

where $\mathbf{K}$ is now a real valued 12×12 square matrix, and $\mathbf{a}^h$ is a real valued 12×1 column matrix whose elements are a_{kn}, $\hat{a}_{kn}$, a'_{kn}, $\hat{a}'_{kn}$ $(k=1,2,3)$. If A_2, A_3 and A_4 are equal to zero, λ_k is also equal to zero. We will now discuss the solution of equations (38.a) and (38.b). Since $\delta_n=0$ and $\delta_n=1$ are the roots of the equation(39)[7], a solution $\mathbf{a}^{P_1}$ and $\mathbf{a}^{P_2}$ in equations (38.a) and (38.b) exists if and only if

$$\mathbf{s}_1\cdot\mathbf{b}_1=0 \quad \cdots\cdots (40.a)$$

$$\mathbf{s}_2 \cdot (A_2\mathbf{b}_2+A_3\mathbf{b}_3+A_4\mathbf{b}_4)=0 \quad \cdots\cdots\cdots\cdots\cdots \quad (40.b)$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are an eigenvector of $\mathbf{K}^T(0)$ and $\mathbf{K}^T(1)$, respectively,

$$\mathbf{K}^T(0)\cdot\mathbf{s}_1=0, \quad \mathbf{K}^T(1)\cdot\mathbf{s}_2=0 \quad \cdots\cdots\cdots \quad (41.a,b)$$

3. Numerical Results and Discussion

In this section, we choose one typical material for example to examine the homogeneous solution for delamination cracks, and numerically confirm the power type eigenfunction by checking the algebraic and geometric multiplicity. Next we examine a particular solution and check whether the polynomial type particular solution exists or not.

For the purpose of illustrating a particular solution, we consider a case wherein each of the loading parameters A_1, A_2, A_3 and A_4 is applied alone, respectively. A solution for a general loading may be obtained from a linear combination of the solutions for these four cases. The special cases of $A_2\mathbf{b}_2+A_3\mathbf{b}_3+A_4\mathbf{b}_4=0$ are excluded here. In each case, equations (40.a,b) and (41.a,b) are then rewritten as follows

$$\mathbf{s}_1\cdot\mathbf{b}_1=0 \;,\; \mathbf{s}_2\cdot\mathbf{b}_2=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (42.a,b)$$
$$\mathbf{s}_2\cdot\mathbf{b}_3=0 \;,\; \mathbf{s}_2\cdot\mathbf{b}_4=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (42.c,d)$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are eigenvectors of $\mathbf{K}^T(0)$ and $\mathbf{K}^T(1)$,

$$\mathbf{K}^T(0)\cdot\mathbf{s}_1=0, \quad \mathbf{K}^T(1)\cdot\mathbf{s}_2=0 \quad \cdots\cdots\cdots \quad (43.a,b)$$

We recall that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ and $\mathbf{b}_4$ are column matrices related to loading parameters A_1, A_2, A_3 and A_4, respectively.

For numerical computation, we use the following material data for the graphite epoxy T300/5208[9]

$$E_L = 134\text{GPa}, \quad E_T = E_Z = 10.2\text{GPa}$$
$$G_{LT} = G_{LZ} = 5.52\text{GPa}, \quad G_{TZ} = 3.43\text{GPa}$$
$$\nu_{LT} = \nu_{LZ} = 0.3 \;, \quad \nu_{TZ} = 0.49 \quad \cdots\cdots\cdots\cdots \quad (44)$$

Using this data, the eigenvalues μ_k $(k=1\sim6)$ and the associated eigenvectors v_{ik} are obtained from equations (11) and (10.a). The values of τ_{ijk} and λ_k are then obtained from equations (15.a) and (8.a). From these results, we can finally calculate $\mathbf{K}(0)$, $\mathbf{K}(1)$, $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ and $\mathbf{b}_4$. The major eigenvalues are tabulated in Table 1 for each of the opened and the closed cracks.

Table 1. Eigenvalues δ_n in homogeneous solution $(n=0,1,2,3\cdots\cdots)$

	eigenvalues
opened crack	$n-1/2$ (single root) n (triple roots) $(n-1/2)\pm i\gamma$ (single root)
closed crack	$n-1/2$ (double roots) n (quadruple roots)

γ : oscillatory index
$i=\sqrt{-1}$

From the computational results, it turns out that for all of the ply orientations, there are three eigenvectors of $\mathbf{K}(0)$ and $\mathbf{K}(1)$ for the opened crack problem, respectively, while there are four eigenvectors of $\mathbf{K}(0)$ and $\mathbf{K}(1)$ for the closed crack problem, respectively. The algebraic multiplicity for each case is 3 and 4, respectively. Thus it appears that the existence of logarithmic homogeneous solution is excluded for either case, and the assumed power type eigenfunction equation (16) is valid.

In order to check the consistency conditions (42.a$\sim$d), the three eigenvectors for $\mathbf{K}^T(0)$ and $\mathbf{K}^T(1)$, are calculated for an opened crack, and similarly the four eigenvectors for a closed crack. For each of these two cases, it turns out that the consistency conditions(42.a$\sim$d) are all satisfied by every set of an eigenvector regardless of any combination of ply orientations. This means that the power type particular solutions exist for all of the fiber angles under typical loadings such as extension, bending and torsion. Therefore, for an opened crack problem equation (38.a) has a particular solution $\mathbf{a}^{(P_1)}$ and three arbitrary solution $\mathbf{a}^{(m)}(m=1\sim3)$, and equation (38.b) has a particular solution $\mathbf{a}^{(P_2)}$ and three arbitrary solution $\mathbf{a}^{(n)}$ (n=4~6). On the other hand, for a closed crack problem, equation (38.a) has a particular solution $\mathbf{a}^{(P_1)}$ and four arbitrary solution $\mathbf{a}^{(m)}$(m =1~4), and equation(38.b) has a particular solution $\mathbf{a}^{(P_2)}$ and four arbitrary solution $\mathbf{a}^{(n)}$ (n=5 ~8).

$$\mathbf{a}^{P} = \mathbf{a}^{P_1} + \mathbf{a}^{P_2} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (45)$$

where

$$\begin{aligned} \mathbf{a}^{P_1} &= \mathbf{a}^{(P_1)} + \alpha_1\mathbf{a}^{(1)} + \alpha_2\mathbf{a}^{(2)} + \alpha_3\mathbf{a}^{(3)} \\ \mathbf{a}^{P_2} &= \mathbf{a}^{(P_2)} + \alpha_4\mathbf{a}^{(4)} + \alpha_5\mathbf{a}^{(5)} + \alpha_6\mathbf{a}^{(6)} \end{aligned}$$

if crack faces are opened

$$\begin{aligned} \mathbf{a}^{P_1} &= \mathbf{a}^{(P_1)} + \alpha_1\mathbf{a}^{(1)} + \alpha_2\mathbf{a}^{(2)} + \alpha_3\mathbf{a}^{(3)} + \alpha_4\mathbf{a}^{(4)} \\ \mathbf{a}^{P_2} &= \mathbf{a}^{(P_2)} + \alpha_5\mathbf{a}^{(5)} + \alpha_6\mathbf{a}^{(6)} + \alpha_7\mathbf{a}^{(7)} + \alpha_8\mathbf{a}^{(8)} \end{aligned}$$

if crack faces are in frictionless contact

where $\mathbf{a}^{(P_1)}$ and $\mathbf{a}^{(P_2)}$ are the solution vector of smallest length $|\mathbf{a}^{P_1}|$ and $|\mathbf{a}^{P_2}|$, respectively, and α_i are arbitrary constants, and $\mathbf{a}^{(m)}$ are the homogeneous solutions in equation (38.a) and $\mathbf{a}^{(n)}$ are the homogeneous solutions in equation (38.b). Substituting equation (45) into equations (34) and (35) with $\delta_n=0$ and $\delta_n=1$, we obtain the particular solution as

$$u_i^P = u_i^{(P_1)} + u_i^{(P_2)} + \sum_{k=1}^{3}\alpha_k u_i^{(k,0)} + \sum_{k=4}^{6}\alpha_k u_i^{(k,1)} \quad \cdots\cdots\cdots \quad (46)$$

$$\sigma_{ij}^P = \sigma_{ij}^{(P_1)} + \sigma_{ij}^{(P_2)} + \sum_{k=1}^{3}\alpha_k \sigma_{ij}^{(k,0)} + \sum_{k=4}^{6}\alpha_k \sigma_{ij}^{(k,1)}$$

if crack faces are opened ··· (47)

$$u_i^P = u_i^{(P_1)} + u_i^{(P_2)} + \sum_{k=1}^{4}\alpha_k u_i^{(k,0)} + \sum_{k=5}^{8}\alpha_k u_i^{(k,1)} \quad \cdots\cdots\cdots \quad (48)$$

$$\sigma_{ij}^P = \sigma_{ij}^{(P_1)} + \sigma_{ij}^{(P_2)} + \sum_{k=1}^{4}\alpha_k \sigma_{ij}^{(k,0)} + \sum_{k=5}^{8}\alpha_k \sigma_{ij}^{(k,1)}$$

if crack faces are in frictionless contact ··· (49)

where

$$u_i^{(P_1)} = \sum_{k=1}^{3}[a_{k0}^{(P_1)}\mathrm{Re}(v_{ik}z_k) + \hat{a}_{k0}^{(P_1)}\mathrm{Im}(v_{ik}z_k)] + \delta_{i3}A_1x_3$$

$$u_i^{(m,0)} = \sum_{k=1}^{3}[a_{k0}^{(m)}\mathrm{Re}(v_{ik}z_k) + \hat{a}_{k0}^{(m)}\mathrm{Re}(v_{ik}z_k)]$$

$$\sigma_{ij}^{(P_1)} = \sum_{k=1}^{3}[a_{k0}^{(P_1)}\mathrm{Re}(\tau_{ijk}) + \hat{a}_{k0}^{(P_1)}\mathrm{Im}(\tau_{ijk})] + C_{ij33}A_1$$

$$\sigma_{ij}^{(m,0)} = \sum_{k=1}^{3}[a_{k0}^{(m)}\mathrm{Re}(\tau_{ijk}) + \hat{a}_{k0}^{(m)}\mathrm{Im}(\tau_{ijk})]$$

$$\begin{cases} m=1\sim3 \text{ for an opened crack} \\ m=1\sim4 \text{ for a closed crack} \end{cases}$$

$$u_i^{(P_2)} = \sum_{k=1}^{3}[a_{k1}^{(P_2)}\mathrm{Re}(v_{ik}z_k^2) + \hat{a}_{k1}^{(P_2)}\mathrm{Im}(v_{ik}z_k^2)]/2 + u_i^*(x_1,x_2,x_3) - \delta_{i3}A_1x_3$$

$$u_i^{(n,1)} = \sum_{k=1}^{3}[a_{k1}^{(n)}\mathrm{Re}(v_{ik}z_k^2) + \hat{a}_{k1}^{(n)}\mathrm{Re}(v_{ik}z_k^2)]/2$$

$$\sigma_{ij}^{(P_2)} = \sum_{k=1}^{3}[a_{k1}^{(P_2)}\mathrm{Re}(\tau_{ijk}z_k) + \hat{a}_{k1}^{(P_2)}\mathrm{Im}(\tau_{ijk}z_k)]/2 + \sigma_{ij}^*(x_1,x_2) - C_{ij33}A_1$$

$$\sigma_{ij}^{(n,1)} = \sum_{k=1}^{3}[a_{k1}^{(n)}\mathrm{Re}(\tau_{ijk}z_k) + \hat{a}_{k1}^{(n)}\mathrm{Im}(\tau_{ijk}z_k)]$$

$$\begin{cases} n=4\sim6 \text{ for an opened crack} \\ n=5\sim8 \text{ for a closed crack} \end{cases}$$

So far we have used Lekhnitskii's formulation and Stroh formalism to determine the structure of the complete solution and investigate the existence of the polynomial type particular solutions. From equations (46) and (47) or equations (48) and (49), it turns out that the polynomial type particular solutions for displacement and stress take quadratic and linear form, respectively. These equations are consistent with general forms of the particular solutions for uniform extension which was studied by Wang[3] based upon

Lekhnitskii's complex potentials.

In this example, no logarithmic solutions appear in the homogeneous solution, neither in the particular solution, regardless of ply orientations. Although numerical verification is limited to one example of material here, our experience shows that this result is valid for most of the composite materials including graphite epoxy, boron epoxy. If one want to confirm this, he may follow foregoing procedure to verify this for a specific material.

This asymptotic solution may be combined with some numerical technique such as the boundary collocation method[2] or the singular hybrid F.E.M.[10]. Without knowing the asymptotic solution structure a priori, particularly without knowing whether there exists any logarithmic singularity or not, one cannot expect his numerical results to be accurate enough. This was illustrated for free edge problem by Stolarski and Chiang [11]. The same would go for the delamination crack problem.

4. Conclusions

The asymptotic solution (homogeneous and particular solutions) for delamination crack in a laminated composite strip undergoing a generalized plane deformation under extension, bending and torsion was examined based upon Lekhnitskii's formulation and Stroh formalism. For both of the opened and the closed, no logarithmic solutions appear in either of homogeneous and particular solutions, regardless of ply orientations. Thus the power type eigenfunction expansion is valid for a homogeneous solution, and the particular solution takes a polynomial function for the loadings considered here-extension, bending and torsion, whether the crack faces are opened or closed.

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