論文

Vibration Analysis of Special Orthotropic Plates on Elastic Foundation with Arbitrary Boundaries

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자유경계를 갖고 탄성기초에 놓인 특별직교이방성 적층복합판의 진동해석

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초 록

불규칙단면과 다양한 경계조건을 갖는 보와 타워구조물의 1모드 하에서 고유진동수를 계산하는 방법은 1974년 Kim, D.H에 의해 개발되었다. 이 논문에서는 자유경계를 갖고 탄성지지된 특별직교 이방성 적층복합판에 대한 이 방법의 응용결과가 주어진다. 이러한 판들은 고속도로 콘크리트 슬래브와 교량 혼성 복합 포장에서 볼 수 있다. 이 진동해석을 위해 어떤 방법이라도 처짐에 대한 영향면을 구하는데 사용될 수 있다. 이 논문에서는 유한차분법이 사용되었다. 고유진동수에 대한 기초 탄성계수의 영향과 판의 형상비의 영향이 연구되었다. 고유진동수에 대한 판 자중의 무시효과가 또한 심도있게 연구되었다.

ABSTRACT

A method of calculating the natural frequency corresponding to the first mode of vibration of beams and tower structures, with irregular cross sections and with arbitrary boundary conditions was developed and reported by Kim, D. H. in 1974. In this paper, the result of application of this method to the special orthotropic plates on elastic foundation with free boundaries is presented. Such plates represent the concrete highway slab and hybrid composite pavement on bridges. Any method may be used to obtain the deflection influence surfaces needed for this vibration analysis. Finite difference method is used for this purpose, in this paper. The influence of the modulus of the foundation and the aspect ratio of the plate on the natural frequency is thoroughly studied. The effect of neglecting the mass of the plates on the natural frequency, as the ratio of the point mass/masses to the plate mass increases, is also studied, in deep.

1. INTRODUCTION

The problem of deteriorated highway concrete slab is very serious all over the world. Before making any decision on repair work, reliable non-destructive evaluation is necessary. One of the

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dependable methods is to evaluate the in-situ stiffness of the slab by means of obtaining the natural frequency. By comparing the in-situ stiffness with the one obtained at the design stage, the degree of damage can be estimated rather accurately.

The reinforced concrete slab can be assumed as a special orthotropic plate, as a close approximation. The highway slab is supported on elastic foundation, with free boundaries. Sometimes, the pair of edges perpendicular to the traffic direction may be subject to the axial forces, even though this problem is not treated in this paper.

Several materials should be tested to find out the best type of pavement material for the future bridge decks, especially advanced composite bridge decks. One solution can be a combination of impregnated woven fibers and toughened polymer with little abrasion property, forming an integral section with composite deck part. Such pavement will behave as the special orthotropic plate on elastic foundation with free edges.

Such plates are subject to the concentrated mass/masses in the form of traffic loads, or the test equipments such as accelerator in addition to their own masses. Analysis of such problem is usually very difficult.

The special orthotropic plate with boundary conditions other than Navier or Levy solution types, or with irregular cross section, or with nonuniform mass including point masses, analytical solution is very difficult to obtain. Numerical method for engenvalue problems are also very much involved in seeking such a solution[6,8].

Most of the civil and architectural structures are large in sizes and the number of laminae is large, even though the thickness to length ratio is small enough to allow to neglect the transverse shear deformation effect in stress analysis. For such plates, there are enough number of fiber orientations for which theories for special orthotropic plates can be applied, and simple formulas developed by Kim, D. H. can be used[5,6,9,10].

However, if the plate has boundary condition other than simple supported, obtaining a reliable solution is very difficult. The basic concept of the Rayleigh method, the most popular analytical method for vibration analysis of a single degree of freedom system, is the principle of conservation of energy ; the energy in a free vibrating system must remain constant if no damping forces act to absorb it. In case of a beam, which has an infinite number of degrees of freedom, it is necessary to assume a shape function in order to reduce the beam to a single degree of freedom system[11]. The frequency of vibration can be found by equating the maximum strain energy developed during the motion to the maximum kinetic energy. This method, however yields the solution either equal to or larger than the real one. For a complex beam, assuming a correct shape function is not possible. In such cases, the solution obtained is larger than the real one. Recall that Rayleigh's quotient ≥ 1 [6] (pp. 189-191).

A simple but exact method of calculating the natural frequency corresponding to the first mode of vibration of beam and tower structures with irregular cross-sections and attached mass/masses was developed and reported by Kim, D. H. in 1974[4]. Recently, this method was extended to two dimensional problems including composite laminates, and has been applied to composite plates with various boundary conditions with/without shear deformation effects and reported at several international conferences including the Eighth Structures Congress and Fourth materials congress of American Society of Civil Engineers.

The merit of the presented method is that it uses such influence coefficient values, used already for calculating deflection, slope, moment and shear to obtain the natural frequency of the structure. When the plate has concentrated mass or masses, one can simply add these masses to the plate mass and use the same deflection influence surfaces to obtain the natural frequency.

In this paper, the result of application of this method to the subject problem is presented. The effect of concentrated point mass/masses is also studied.

2. METHOD OF ANALYSIS

In this paper, the method of analysis given in detail, in the Kim's book[6,7] is repeated. The magnitudes of the maximum deflection at a certain number of points are arbitrarily given as

$$w(i,j)(1) = W(i,j)(1) \tag{1}$$

where (i,j) denotes the point under consideration. This is absolutely arbitrary but educated guessing is good for accelerating convergence. The dynamic force corresponding to this(maximum) amplitude is

$$F(i,j)(1) = m(i,j)[\omega(i,j)(1)]^2\, w(i,j)(1) \tag{2}$$

The "new" deflection caused by this force is a function of F and can be expressed as

$$\begin{aligned} w(i,j)(2) &= f\{m(k,l)[\omega(i,j)(1)]^2\, w(k,l)(1)\} \\ &= \sum^{k,l} \Delta(i,j,k,l)\{m(k,l)[\omega(i,j)(1)]^2 w(k,l)(1)\} \end{aligned} \tag{3}$$

where Δ is the deflection influence surface. The relative (maximum) deflections at each point under consideration of a structural member under resonance condition, w(i,j)(1) and w(i,j)(2), have to remain unchanged and the following condition has to be held :

$$w(i,j)(1)/w(i,j)(2) = 1 \tag{4}$$

From this equation, $\omega(i,j)(1)$ at each point of (i,j) can be obtained. But they are not equal in most cases. Since the natural frequency of a structural member has to be equal at all points of the member, i.e., $\omega(i,j)$ should be equal for all (i,j), this step is repeated until sufficient equal magnitude of $\omega(i,j)$ is obtained at all (i,j) points.

However, in most cases, the difference between the maximum and the minimum values of $\omega(i,j)$ obtained by the first cycle of calculation is sufficiently negligible for engineering purposes. The accuracy can be improved by simply taking the average of the maximum and the minimum, or by taking the value of $\omega(i,j)$ where the deflection is the maximum. For the second cycle, w(i,j)(2) in

$$w(i,j)(3) = f\{m(i,j)[\omega(i,j)(2)]^2\, w(i,j)(2)\} \tag{5}$$

the absolute numerics of w(i,j)(2) can be used for convenience.

3. NUMERICAL EXAMINATION

$[A/B/A]_r$ type laminate is considered. The material properties are ;

$E_1 = 67.36$ GPa,	$E_2 = 8.12$ GPa,
$\nu_{12} = 0.272$,	$\nu_{21} = 0.0328$,
$G_{12} = 3.0217$ GPa,	$h = 0.015$ m

The thickness of a ply is 0.005m. As the r increases, B_{16}, B_{26}, D_{16} and D_{26} decreases and the equations for the special orthotropic plates can be used. For simplicity, it is assumed that A=0°, B=90°, and r=1. Then D(1,1)=18492.902 Nm.

3.1 Finite difference method (F.D.M.)

The method used in this paper requires the deflection influence surfaces. Since no reliable analytical method is available for the subject problem, F.D.M. is applied to the governing equation of the special orthotropic plates,

$$D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4} = q(x, y)-kw \tag{6}$$

where $D_1=D_{11}$, $D_2=D_{22}$, $D_3=(D_{12}+2D_{66})$.

The number of the pivotal points required, in the case of the order of error Δ^2, where Δ is the mesh size, is five for the central differences. This makes the procedure at the boundaries complicated. In order to solve such problem, the three simultaneous partial differential equations of equilibrium with three dependent variables, ω, M_x, M_y, are used instead of equation(6) [1-3].

$$\frac{\partial^2 M_x}{\partial x^2}+4D_{66}\frac{\partial^4 w}{\partial_x^2\partial y^2}+\frac{\partial M_y}{\partial y^2}$$
$$=-q(x,y)+kw(x,y) \tag{7}$$

$$M_x=-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2} \tag{8}$$

$$M_y=-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial_y} \tag{9}$$

If F.D.M. is applied to these equations, the resulting matrix equation is very large in sizes, but the tridiagonal matrix calculation scheme used by Kim[1-3] is very efficient to solve such equations.

3.2 Accuracy of F.D.M

Since one of the few efficient analytical solutions of the special orthotropic plate is Navier solution, and this is good for the case of the four edges simple supported, F.D.M. is used to solve this problem and the result is compared with the Navier solution.

The aspect ratio used is 1m/1m=1. The mesh size is Δx =0.1, Δy = 0.1. The deflection at (x,y), the origin of the coordinates being the corner of the plate, is obtained, and the ratio of the Navier solution to the F.D.M. solution is given in Table 1.

Calculation is carried out with different mesh sizes and the maximum errors at the center of the plate(loading point) are as follows. Loading is 100N/m^2 at the center of the plate.

10×10 case : 0.14%

20×20 case : 0.035%

40×40 case : 0.009%

It should be noted that comparison is made at the loading point and both methods may have some errors involved. At most of the other points, the error is less than 1% even when mesh size is $0.1\times0.1(10\times10)$, as predicted.

3.3 Influence of the modulus of foundation and aspect ratio of the plate

For this study, the simplest assumption that the intensity of the reaction of the subgrade is proportional to the deflection, w, of the plate, is used. Thus, this intensity is given by the expression kw, where k is the constant called the modulus of the foundation, in Newton per square meter per meter of deflection12). The plate geometry and applied

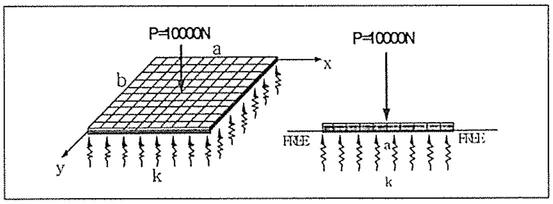


Fig. 1. Plate geometry and loading

Table 1. Deflection ratio of Navier solution to F.D.M. solution

	Navier / F.D.M.				
x(m) / y(m)	0.1	0.3	0.5	0.7	0.9
0.1	0.1005946E+01	0.1004916E+01	0.1004713E+01	0.1004916E+01	0.1005946E+01
0.3	0.1001279E+01	0.1000028E+01	0.9996814E+00	0.1000028E+01	0.1001279E+01
0.5	0.1000134E+01	0.9989528E+00	0.9985780E+00	0.9989530E+00	0.1000134E+01
0.7	0.1001279E+01	0.1000028E+01	0.9996815E+00	0.1000028E+01	0.1001279E+01
0.9	0.1005946E+01	0.1004916E+01	0.1004714E+01	0.1004916E+01	0.1005946E+01

load is as given in Figure 1.

The laminate is $[0^\circ, 90^\circ, 0^\circ]_5$, with same material properties as given previously.

The effect of the modulus of foundation, k, and the plate aspect ratio(a/b) on the deflection at the loading point is given in Table 2. Figure 2 show the same result as Table 2, in graphics.

Vibration analysis is carried out by the method presented in this paper. The result is given in Table 3. Figure 3 is the graphic presentation of Table 3.

3.4 The effect of concentrated mass

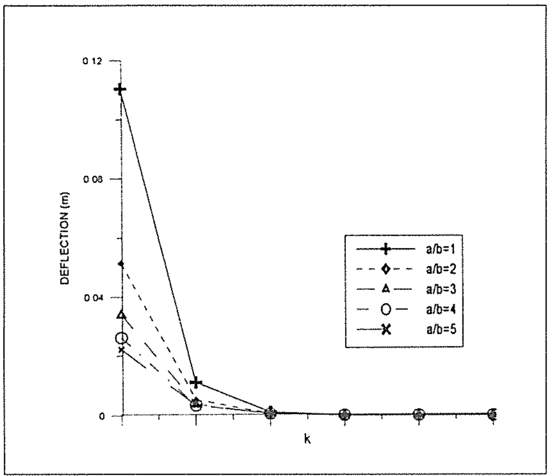


Fig. 2. Deflection at the load point (center of the plate) with different aspect ratios and k values

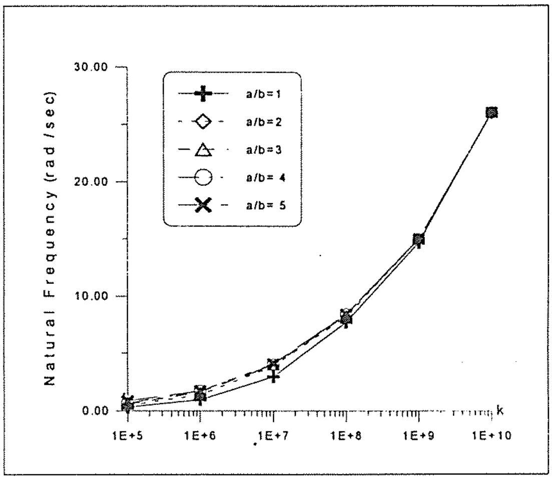


Fig. 3. Natural Frequencies for each case of k and a/b

Table 2. Deflection at the loading point (center of the plate)

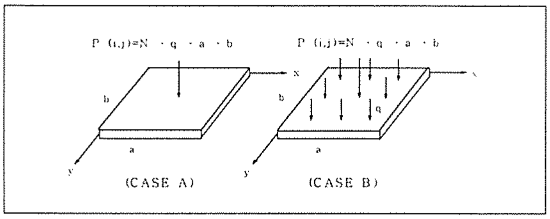
(k : modulus of foundation[12], Unit : m)

k(N/m³) / Aspect Ratio	$k = 10^5$	$k = 10^6$	$k = 10^7$	$k = 10^8$	$k = 10^9$	$k=10^{10}$
1	0.11033E+00	0.11091E−01	0.11665E−02	0.16834E−03	0.46369E−04	0.14758E−04
2	0.51433E−01	0.53099E−02	0.67902E−03	0.14884E−03	0.44541E−04	0.14693E−04
3	0.33893E−01	0.38341E−02	0.64557E−03	0.14333E−03	0.44461E−04	0.14693E−04
4	0.26165E−01	0.34579E−02	0.63320E−03	0.14163E−03	0.44459E−04	0.14693E−04
5	0.22175E−01	0.34139E−02	0.61221E−03	0.14156E−03	0.44459E−04	0.14693E−04

Table 3. Natural frequencies for each case of k and aspect ratio, a/b.

(Unit : rad/sec)

k(N/m³) / Aspect Ratio	$k = 10^5$	$k = 10^6$	$k = 10^7$	$k = 10^8$	$k = 10^9$	$k=10^{10}$
1	0.30105E+00	0.94950E+00	0.29278E+01	0.77072E+01	0.14685E+02	0.26031E+02
2	0.44099E+00	0.13722E+01	0.38374E+01	0.81965E+01	0.14984E+02	0.26088E+02
3	0.54310E+00	0.16148E+01	0.39355E+01	0.83526E+01	0.14997E+02	0.26088E+02
4	0.61809E+00	0.17004E+01	0.39738E+01	0.84026E+01	0.14997E+02	0.26088E+02
5	0.82750E+00	0.17119E+01	0.40413E+01	0.84047E+01	0.14997E+02	0.26088E+02



a=b=1m, k=10MN/m^3, q=1433.25N/m^2, [0°, 90°,0°]$_5$
Fig. 4. Loading conditions of a plate

Figure 4. show the loading conditions of a plate with [0°, 90°,0°]$_5$ orientation, with a=b=1m, k=10MN/m^3, and q=1433.25N/m^2. N is a real number. P(i,j) is at the center of the plate. Tables 4, 5 and 6 show the deflection of the CASE B with N=0, CASE A when N=1, and CASE B when N=1, respectively.

Table 7. show the deflection at the center of the plate with different values of N. Table 8 show the natural frequency of the plate with different values of N.

4. CONCLUSION

In this paper, the simple and accurate method of vibration analysis developed by D. H. Kim is presented. The presented method is simple to use but extremely accurate. The boundary condition can

Table 4. Deflection of the plate, CASE B, N=0

(Unit : m)

y(m) \ x(m)	0.1	0.3	0.5	0.7	0.9
0.1	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03
0.3	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03
0.5	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03
0.7	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03
0.9	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03	0.1418E−03

Table 5. Deflection of the plate, CASE A, N=1

(Unit : m)

y(m) \ x(m)	0.1	0.3	0.5	0.7	0.9
0.1	0.15389E−03	0.15476E−03	0.15511E−03	0.15476E−03	0.15389E−03
0.3	0.15712E−03	0.16046E−03	0.16206E−03	0.16046E−03	0.15712E−03
0.5	0.15847E−03	0.16342E−03	0.16718E−03	0.16342E−03	0.15847E−03
0.7	0.15712E−03	0.16046E−03	0.16206E−03	0.16046E−03	0.15712E−03
0.9	0.15389E−03	0.15476E−03	0.15511E−03	0.15476E−03	0.15389E−03

Table 6. Deflection of the plate, CASE B, N=1

(Unit : m)

y(m) \ x(m)	0.1	0.3	0.5	0.7	0.9
0.1	0.29567E−03	0.29654E−03	0.29688E−03	0.29654E−03	0.29567E−03
0.3	0.29888E−03	0.30218E−03	0.30376E−03	0.30218E−03	0.29888E−03
0.5	0.30021E−03	0.30511E−03	0.30883E−03	0.30511E−03	0.30021E−03
0.7	0.29888E−03	0.30218E−03	0.30376E−03	0.30218E−03	0.29888E−03
0.9	0.29567E−03	0.29654E−03	0.29688E−03	0.29654E−03	0.29567E−03

Table 7. Deflection at the center of the plate (Unit : m)

N	Case A	Case B	Case A/ Case B
0		0.1418E−03	
1	0.16718E−03	0.30883E−03	0.54133
3	0.50155E−03	0.64321E−03	0.77976
5	0.83593E−03	0.97758E−03	0.85510
10	0.16718E−02	0.18135E−02	0.92186
20	0.33437E−02	0.34853E−02	0.95937
40	0.66874E−02	0.68291E−02	0.97925

Table 8. Natural frequency (rad/sec)

N	Case A	Case B	Case A/ Case B
0		0.81437E+01	
1	0.77317E+01	0.55498E+01	0.71780
3	0.44647E+01	0.38865E+01	0.87050
5	0.34585E+01	0.31664E+01	0.91554
10	0.24456E+01	0.23349E+01	0.95474
20	0.17293E+01	0.16887E+01	0.97652
40	0.12228E+01	0.12081E+01	0.98798

be arbitrary. Both stiffness and mass of the element can be variable. One can use any method to obtain the deflection influence coefficient. The accuracy of the solution is dependent on only that of the influence coefficients needed for this method. One should recall that obtaining the deflection influence coefficients is the first step in design and analysis of a structure. The merit of the presented method is that it uses such influence coefficient values, used already for calculating deflection, slope, moment and shear to obtain the natural frequency of the structure. When the plate has concentrated mass or masses, one can simply add these masses to the plate mass and use the same deflection influence surfaces to obtain the natural frequency.

This method is applied to the special orthotropic plate on elastic foundation with free boundaries. Such plate is the case of the most of the concrete highway slab and hybrid composite pavement on bridge. Finite difference method is used to obtain the deflection influence surfaces in this paper.

The effect of the modulus of foundation, the aspect ratio of the plate, and the concentrated attached mass on the plate, on the natural frequency is thoroughly studied and the result is given in tables to provide a guideline to the design engineers.

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